M.sc

Open Mapping Theorem & The closed-graph Theorem.

M.sc (Sem) 03, Paper-11, Unit-03

Examples.

Example-(A): Let $\{T_n\}$ be a sequence of continuous linear operators of Banach space X into Banach space Y such that $\lim_{n\to\infty} T_n(x)$ exists for every $x \in X$. Then prove that T is a continuous linear operator and $\|T\| \le \liminf_{n\to\infty} \|T_n\|$

Solution:— 1. T is linear

(a) We have $T(x+y) = \lim_{n\to\infty} T_n(x+y)$. since each T_n is linear.

$$T_n(x+y) = T_n(x) + T_n(y)$$

$$\therefore \lim_{n\to\infty} T_n(x+y) = \lim_{n\to\infty} [T_n(x) + T_n(y)]$$

$$= \lim_{n\to\infty} T_n(x) + \lim_{n\to\infty} T_n(y)$$

$$= T(x) + T(y)$$

or $T(x+y) = T(x) + T(y)$ ...

(b) $T(\alpha x) = \lim_{n\to\infty} T_n(\alpha x)$

$= \lim_{n\to\infty} \alpha T_n(x)$ as each T_n is linear

$$T(\alpha x) = \alpha \lim_{n\to\infty} T_n(x) = \alpha T(x)$$

Thus T is linear.

2. Since $\lim_{n\to\infty} T_n(x) = T(x)$

$$\|\lim_{n\to\infty} T_n(x)\| = \|T(x)\|$$

or $\lim_{n\to\infty} \|T_n(x)\| = \|T(x)\|$

as norm is a continuous function and T is continuous iff only if $x_n \to x$ implies $Tx_n \to Tx$.

Thus $\|T_n(x)\|$ is a bounded sequence in Y.

(The principle of Uniform boundedness), $\|T_n\|$ is bounded sequence in the space $B[X,Y]$.

This implies that

$$\|T(x)\| = \lim_{n\to\infty} \|T_n(x)\| \leq \lim_{n\to\infty} \inf \|T_n\| \, \|x\| \quad \text{---} (1)$$

We know that by the definition of $\|T\|$, we get

$$\|T\| \leq \lim_{n\to\infty} \inf \|T_n\| \, \|x\|$$

So T is bounded and hence continuous as T is linear.

Example-(B) Let X and Y be two Banach spaces and $\{T_n\}$ be a sequence of continuous linear operators. Then the limit $Tx = \lim_{n\to\infty} T_n(x)$ exists for every $x \in X$ iff

1. $\|T_n\| \leq M$ for $n = 1, 2, 3, \ldots$

2. The limit $Tx = \lim_{n\to\infty} T_n(x)$ exists for every element x belonging to a dense subset of X.

Solution:— I Suppose that the limit $Tx = \lim_{n\to\infty} T_n(x)$ exists $\forall\, x \in X$. Then clearly $Tx = \lim_{n\to\infty} T_n(x)$ exists for x belonging to a dense subset of X.

$$\|T_n\| \leq M \text{ for } n = 1, 2, 3, \ldots$$

II. Suppose 1 and (2) hold, then we want to prove that $T(x) = \lim_{n\to\infty} T_n(x)$ exists i.e. for $\epsilon > 0$, $\exists N$ such that $\|T_n(x) - T(x)\| < \epsilon$ for $n > N$.

Let A be a dense subset of X, then for arbitrary $x \in X$ we can find $x' \in A$ such that $\|x - x'\| < \delta$, $\delta > 0$ and arbitrary —(2)

We have $\|T_n(x) - T(x)\| \leq \|T_n(x) - T_n(x')\| + \|T_n(x') - T(x)\|$

—(3)

By condition (2)

$\|T_n(x') - T(x')\| < \epsilon$, for $n > N$ —(4)

as $x' \in A$, a dense subset of X.

Since T_n's are linear.

$\|T_n(x) - T_n(x')\| = \|T_n(x - x')\|$

by condition (1) Equation (2), we have

$\|T_n(x - x')\| \leq \|T_n\| \, \|x - x'\|$

$\leq M\delta \;\forall n$ —(5)

From equations (3), (4) and (5), we have

$\|T_n(x) - T(x)\| \leq M\delta + \epsilon_1 < \epsilon \;\forall n > N$

This proves the desired result.

Example (c):- Show that the principle of uniform boundedness is not valid if X is only a normed space.

Solution:- Let $Y = \mathbb{R}$ and X be the normed space of all polynomials $x = x(t) = \sum_{n=0}^{\infty} a_n t^n$ where $a_n = 0 \;\forall n > N$.

With the norm $\|x\| = \sup_n |a_n|$. X is not a Banach space.

Define $T_n : X \to Y$ as follows

$$T_n(x) = \sum_{k=0}^{n-1} a_k$$

$\|T_n\|$ is not bounded.

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